Mini projet

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1 Affine algebraic geometry over $\mathbb{R}$: ongoing project with Dubouloz

Let us emphasize a common terminological source of confusion about the meaning of what is a *real algebraic variety*. From the point of view of general algebraic geometry, a real variety X is a variety defined over the real numbers, and a morphism is understood as being defined over all the geometric points. In most real algebraic geometry texts however, the algebraic structure considered corresponds to the algebraic structure of a neighbourhood of the real points $X(\mathbb{R})$ in the whole complex variety – or, in other words, the structure of a germ of an algebraic variety defined over $\mathbb{R}$.

From this point of view it is natural to view $X(\mathbb{R})$ as a compact submanifold of the real affine space $\mathbb{R}^n$ defined by real polynomial equations, where n is some natural integer. It is worthy to remark that the topology induced by the embedding of $X(\mathbb{R})$ as a topological submanifold of $\mathbb{R}^n$ is stronger than the Zariski topology. Note that a real affine variety may be compact. Note also that $X(\mathbb{R})$ may not be connected. Likewise, it is natural to say that a map $\psi\colon X(\mathbb{R}) \to Y(\mathbb{R})$ is an *isomorphism* if ψ is induced by a birational map $\Psi\colon X \dashrightarrow Y$ such that Ψ (respectively Ψ^{-1}) is regular at any point of $X(\mathbb{R})$ (respectively of $Y(\mathbb{R})$). In particular, $\psi\colon X(\mathbb{R}) \to Y(\mathbb{R})$ is a diffeomorphism for the $\mathcal{C}^\infty$-structures induced by the embedings in real affine spaces.

We denote by Aut($X(\mathbb{R})$), the group of automorphisms.

Our very first question is to study the structure of the group $Aut(A^2_{\mathbb{R}})$ of automorphisms of the real affine plane.

2 Infinite transitivity, density, group actions : collaborations with Huisman, Blanc, Kuyumzhiyan, Kollar

[J. Huisman, F. Mangolte, The group of automorphisms of a real rational surface is n-transitive, Bulletin of the London Mathematical Society 41, 563-568 (2009).

J. Huisman, F. Mangolte, Automorphisms of real rational surfaces and weighted blow-up singularities, manuscripta mathematica 132, 1-17 (2010).]

[J. Blanc, F. Mangolte, Geometrically rational real conic bundles and very transitive actions, Compositio Mathematica, arXiv :0903.3101[math.AG], sous presse (2010).]

The group of automorphisms of a complex algebraic variety is small : indeed, it is finite in general. Moreover, the group of automorphisms is 3-transitive only if the variety is P^1. On the other hand, it was recently proved [Huisman-Mangolte2009] that for a surface $X(\mathbb{R})$ birational to $P^2_{\mathbb{R}}$, its group of automorphisms acts n-transitively on $X(\mathbb{R})$ for any n. In such situation, we say that the group is infinitely transitive or acts very transitively on $X(\mathbb{R})$. The paper [Blanc-Mangolte2010] determines all real algebraic surfaces $X(\mathbb{R})$ having a group of automorphisms which acts very transitively on $X(\mathbb{R})$. The paper [Huisman-Mangolte2010] studies the infinite transitivity for certain singular surfaces. Namely surfaces which admits only real sandwich singularities of complex type A_k.

[F. Mangolte, K. Kuyumzhiyan, Very transitive actions on real affine suspensions, in preparation]

Arzhantsev-Kuyumzhiyan-Zaidenberg shows that in many situations, the infinite homogeneity of a complex affine variety induces the infinite homogeneity of its iterated suspensions. Namely, if the special automorphism group of an affine variety Y of dimension greater than 2 acts very transitively on the smooth locus of Y and if at every smooth point of Y the tangent space is spanned by the tangent vectors to the orbits of one-parameter subgroups, then every suspension of Y satisfies the same two properties. The proof in such generality is however valid provided that the ground field is algebraically closed. When the ground field is $\mathbb{R}$, it is proved that the same result holds under two restrictions : the smooth locus of Y is connected and the function f is surjective. The aim of the note by F. Mangolte, K. Kuyumzhiyan is to resolve the real case for any Y and any f.

[J. Kollár, F. Mangolte, Cremona transformations and diffeomorphisms of surfaces, Advances in Mathematics 222, 44-61 (2009).]

They show that the action of Cremona transformations on the real points of quadrics exhibits the full complexity of the diffeomorphisms of the sphere, the torus, and of all non-orientable surfaces. The main result says that if S is rational, then Aut(S), the group of algebraic automorphisms, is dense in Diff(S), the group of self-diffeomorphisms of S.

Note that inspired by that paper, Serge Cantat uses an analogous strategy to prove that every permutation of $P^n(K)$, where K is a finite field with odd characteristic, is induced by a birational transformation with no rational indeterminacy point. [Comptes Rendus Mathematique Volume 347, Issues 21-22, November 2009, Pages 1289-1294]

3 Singular surfaces and example of dimension 3 in real algebraic geometry

[F. Catanese, F. Mangolte, Real singular Del Pezzo surfaces and threefolds fibred by rational curves, I, Michigan Mathematical Journal 56, 357-373 (2008).

F. Catanese, F. Mangolte, Real singular Del Pezzo surfaces and threefolds fibred by rational curves, II, Annales Scientifiques de l'Ecole Normale Supérieure 42, 531-557 (2009).]

The Nash conjecture asks which are the topological types of smooth rational algebraic varieties. More generally, one can ask for the topology of rationally connected varieties. The first theorem in this direction is Comessatti's theorem (1914) : if N is a component of the real part $W(\mathbb{R})$ of a real rational surface W (or, of an algebraic surface with $p_g(W) = 0$), then N cannot be simultaneously orientable and of hyperbolic type. In the nonorientable case we have a similar result under the assumption of minimality of the real rational surface W. Let now $W \to X$ be a real smooth projective 3-fold fibred by rational curves. J. Kollár proved that, if $W(\mathbb{R})$ is orientable, then a connected component N of $W(\mathbb{R})$ is essentially either a Seifert fibred manifold or a connected sum of lens spaces. The main Theorem of Catanese-Mangolte, answering in the affirmative three questions of Kollár, gives sharp estimates on the number and the multiplicities of the Seifert fibres and on the number and the torsions of the lens spaces when X is a geometrically rational surface. When N is Seifert fibred over a base orbifold F, this result generalizes Comessatti's theorem on smooth real rational surfaces : F cannot be simultaneously orientable and of hyperbolic type. Furthermore, they show as a surprise that, unlike in Comessatti's theorem, there are examples where F is non orientable, of hyperbolic type, and X is minimal. The main technique they use is to construct Seifert fibrations as projectivized tangent bundles of Du Val surfaces.

4 Goodies

Ongoing collaboration with Jérémy Blanc about birational geometry of real conic bundles. Organizer of the Workshop "Real aspects of affine algebraic geometry" Angers march 2011.

Pour la question proposée par Serge sur la "taille" du groupe des automorphismes des cubiques réelles de $A4$, on peut demander la densité du groupe des automorphismes dans celui des difféomorphismes de la variété réelle. On peut demander aussi si le groupe des automorphismes est infiniment transitif. Ce sont des questions naturelles pour lesquelles, ceci dit, je n'ai aucune idée a priori.